The Mathematics of Returns-Based Style Analysis

One of the most frequently asked questions concerning the mathematics of StyleAdvisor is how exactly we calculate the style attribution coefficients which are displayed in the style map view, the asset allocation view, and the style table. The short answer to this question is very easy: We perform the *returns-based style analysis* that was invented by Stanford professor and Nobel laureate William F. Sharpe. Sharpe's method is explained in detail in his landmark paper

"Determining a Fund's Effective Asset Mix," Investment Management Review, (December 1988), pp. 59 – 69

In this article, I will explain the mathematics of Sharpe's algorithm. As it turns out, a fairly complete and mathematically rigorous description of the algorithm can be given without using a lot of mathematical formalism.

As you will see, Sharpe's formula bears a strong resemblance to classical constrained multivariate regression techniques. That is the reason why we are often asked whether returns-based style analysis is actually identical with constrained multivariate regression analysis, or how the two methods relate to each other. I will answer this question in the second part of this article.

Once I have explained the mathematics of Sharpe's returns-based style analysis, the question arises why it is that the style coefficients that we're calculating with this particular method constitute an adequate description of a manager's effective asset mix. I will address this question in a follow-up article in the next issue of this newsletter. Let me also mention that a more detailed and more formal discussion of the mathematics behind William F. Sharpe's returns-based style analysis will appear in a chapter that I contributed to the upcoming 3rd edition of T. Daniel Coggin's and Frank J. Fabozzi's "Handbook of Equity Style Management," published by John Wiley & Sons.

The Mathematics of Sharpe's Method

Let us recall first that the purpose of returns-based style analysis is to determine a manager's *effective asset mix* with respect to a set of asset classes. This means that we are trying to determine the manager's exposure to changes in the values of the asset classes. To this end, a set of *style coefficients* is calculated, one for each asset class. Each style coefficient represents the exposure of the manager to the respective asset class.

Let M be the manager return series and A_1, A_2, A_3, A_4 the return series of the chosen asset classes, i.e., the style indices. Sharpe's method determines the style attribution coefficients c_1, c_2, c_3, c_4 in such a way that the variance of the series

$$M - (c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4)$$

becomes minimal. Needless to say, one could use any number of asset classes; the number four is chosen here as an example. Referring to the expression $c_1A_1 + c_2A_2+$

$c_3A_3 + c_4A_4$ (i.e., the weighted composite of the asset classes) as the “style benchmark,” this can be rephrased as follows:

The style attribution coefficients are determined in such a way that the variance of the excess return of the manager over the style benchmark becomes minimal.

If one translates the above into a mathematical algorithm, then the problem boils down to performing a certain quadratic optimization. It is not necessary go into any of the gory details here; the italicized statement above is a complete and rigorous description of the mathematics of returns-based style analysis.

Sharpe's original paper constrains the analysis by requiring that all coefficients be between 0 and 1, and that the coefficients add up to 1. One may relax some or all of these constraints. The discussion below is not affected by the type of constraints that are used.

There is just one more mathematical aspect that is worth discussing, and that is the question of the uniqueness of the solution. What if there were two entirely different sets of style coefficients, resulting in two entirely different style benchmarks, and the excess return of the manager over these two different style benchmarks were the same minimal value? Which one of the two sets of style coefficients would StyleADVISOR choose?

The answer is that such a situation can never occur. It can be proved mathematically that there always exists exactly one set of style coefficients such that the excess return of the manager over the corresponding style benchmark is minimal. This proof is far beyond the scope of this article. The following high-level summary of the proof is for the reader with a graduate level mathematics background: Minimizing the variance of excess return of the manager over the style benchmark amounts to finding the shortest distance between a point and a convex set in a certain Euclidean space; it is true in every Euclidean space that this distance is assumed at exactly one point on the convex set.

Returns-based Style Analysis vs. Multivariate Regression

Classical multivariate regression analysis determines a constant α and coefficients r_1, r_2, r_3, r_4 in such a way that the *sum of the squares* of the series

$$M - (\alpha + r_1A_1 + r_2A_2 + r_3A_3 + r_4A_4)$$

is minimized. If the regression is performed with alpha constrained to 0, then the expression above becomes the same that was used in Sharpe's method, but the quantity that gets minimized is different: in Sharpe's method, it is the *variance*, while in regression analysis, it is the *sum of the squares*.

This shows that Sharpe's method and multivariate regression are simply two different methods with different intents. However, there is a mathematical connection between the two. It can be proved (again, the proof is beyond the scope of this article, but I'll be glad to send you the proof if you'd like to see it) that the coefficients that minimize the *variance* of the expression

$$M - (c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4)$$

happen to be the same ones that minimize the *sum of the squares* of the expression

$$M - (\alpha + r_1A_1 + r_2A_2 + r_3A_3 + r_4A_4)$$

Therefore, the following is true:

Performing a returns-based style analysis according to William F. Sharpe's method is equivalent to performing a classical multivariate linear regression with unconstrained alpha and then "dropping the alpha," i.e., considering only the regression coefficients r_1, r_2, r_3, r_4.

It should be clear that this connection between Sharpe's method and classical regression analysis is rather accidental. The original intent of the two methods is different: minimizing variance is different from minimizing the sum of the squares. It just so happens that under certain circumstances (unconstrained alpha), the coefficients come out to be the same.

Summary

William F. Sharpe's method of returns-based style analysis is substantially different from classical multivariate regression analysis. While there is a strong mathematical connection between Sharpe's method and classical multivariate constrained regression, the two are clearly different. Sharpe's method employs quadratic optimization to minimize the *variance of the excess return* of the manager over a linear combination of the asset classes. Regression analysis, by contrast, seeks to minimize the *sum of the squares* of the difference between the manager and a linear combination of the asset classes. Moreover, the linear combination of the asset classes used in regression analysis usually includes a constant alpha, which is *not* present in Sharpe's method.

In the next issue of this newsletter, I will discuss why William F. Sharpe’s method is an appropriate and meaningful way of determining a manager’s effective asset mix.

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August, 2002